Task 1 (40 points)
Consider the rotation matrix $\mathbb{R}(\theta)$ in two dimension and the projector matrix $\mathbb{P}_x$,

$$\mathbb{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \mathbb{P}_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{1}$$

Calculate the commutator

$$[\mathbb{R}(\theta), \mathbb{P}_x] = \mathbb{R}(\theta)\,\mathbb{P}_x - \mathbb{P}_x\,\mathbb{R}(\theta)\,, \tag{2}$$

and interpret the result geometrically.

Task 2 (40 points)
Consider the matrix representation of a complex number $z = x + \mathrm{i}y$,

$$\mathbb{M}(z) = \begin{pmatrix} \mathrm{Re}\,z & -\mathrm{Im}z \\ \mathrm{Im}\,z & \mathrm{Re}z \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}. \tag{3}$$

Calculate $\mathbb{M}(z^{-1})$ i.e., the matrix representation of the inverse of a complex number, by expression $z^{-1} = z^*/|z|^2$ in terms of x and y, and using Eq. (3). Then, show that

$$\mathbb{M}(z^{-1}) = [\mathbb{M}(z)]^{-1}\,, \tag{4}$$

i.e., that the matrix representation of the inverse of a complex number z is equal to the inverse matrix of the matrix representation of z.

Task 3 (10 points)
Easy exercise: Show that $\mathbb{M}(z_1 + z_2) = \mathbb{M}(z_1) + \mathbb{M}(z_2)$, and that $\mathbb{M}(z_1\, z_2) = \mathbb{M}(z_1)\mathbb{M}(z_2)$.

Task 4 (50 points)
Consider the following complex paths (complex integration contours),

- contour C_1: a straight line in the complex plane from the origin to the point (3,3),
- contour C_2: the curve $z(t) = 3t + 3\mathrm{i}t^2$ with $t \in (0, 1)$,
- contour C_3: the straight line from 0 to 3i and then another straight line from 3i to $3 + 3\mathrm{i}$,

All of the above paths (contours) start at $z_1 = 0$ and end at $z_2 = 3 + 3\mathrm{i}$. Make three drawings showing all of the above indicated paths in the complex plane. Evaluate all three integrals

$$I_i = \int_{C_i} z^2 \mathrm{d}z\,, \qquad i = 1, 2, 3\,, \tag{5}$$

using the formalism introduced in the lecture. *Evaluate all integrals explicitly along their given contours. Just using a general argument to evaluate all of them will not(!) address the question and will not(!) give you any points!* Compare your results with what you otherwise obtain by evaluating the expression $I = (z_2^3 - z_1^3)/3$.

Task 5 (30 points)
Consider the function $f(z) = f(x + \mathrm{i}y) = \mathrm{e}^z \cos^2(z) = f_1(x, y) + \mathrm{i}f_2(x, y)$. Write the real and imaginary parts of $f(z)$ (i.e., the functions f_1 and f_2) as a function of x and y. Show that

$$\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} = 0\,, \qquad \frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y} = 0\,. \tag{6}$$

Interpret your results in terms of the divergence of the vector fields $f_1\,\hat{\mathrm{e}}_x - f_2\hat{\mathrm{e}}_y$ and $f_2\,\hat{\mathrm{e}}_x + f_1\hat{\mathrm{e}}_y$.

The tasks are due Thursday, 29–JAN–2020.