Task 1 (40 points)
Associate the complex function

$$f(z) = f_1(x,y) + \mathrm{i}\, f_2(x,y)\,, \qquad z = x + \mathrm{i}y\,, \tag{1}$$

with the vector field

$$\vec{F}^*(x,y) = f_1(x,y)\,\hat{e}_x - f_2(x,y)\,\hat{e}_y\,. \tag{2}$$

In which sense is the imaginary part of a complex contour integral of $f(z)$ around a closed contour related to the divergence of $\vec{F}^*$? Derive a corresponding relation and argue, by pointing out a particular plot in the notes, that the function $f(z) = 1/z$ might be special in this regard. *You may use lecture notes.*

Task 2 (40 points)
For a complex function

$$f(z) = f_1(x,y) + \mathrm{i}\, f_2(x,y)\,, \qquad z = x + \mathrm{i}y\,, \tag{3}$$

we have derived the Cauchy–Riemann differential equations

$$\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} = 0\,, \qquad \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} = 0\,. \tag{4}$$

Now consider a function $f = f(z^*)$ as opposed to $f = f(z)$. *Derive the analogue of the Cauchy–Riemann differential equations for a function $f = f(z^*)$.*

Task 3 (40 points)
Evaluate the complex contour integral

$$\int_{C(a,b)} \frac{1}{z}\,\mathrm{d}z \tag{5}$$

where $C(a,b)$ is a closed contour, centered at the origin, around an ellipsoid with half axes a (in the x direction) and b (in the y direction). *Use the contour as indicated in the task. Just quoting general results, even if correct, will result in zero credit. Again, use the contour as indicated in the task.*

Hint: You *might* proceed as follows. Find a suitable parameterization of the ellipsoid in the complex plane. Parameterize a point on the ellipsoid by an angle θ. For example, try an ansatz $z(\theta) = a\cos(\theta) + \mathrm{i}b\sin(\theta)$. Differentiate with respect to θ to get $\mathrm{d}z$. Then, integrate the function.

The tasks are due Thursday, 11–FEB–2021.