Task 1 (40 points)
Consider a two-dimensional vector

$$\vec{r} = x\,\hat{e}_x + y\,\hat{e}_y\,. \tag{1}$$

Consider the mapping

$$\vec{r} \to \vec{r}\,' = \vec{r} \times \hat{e}_z = x'\,\hat{e}_x + y'\,\hat{e}_y\,. \tag{2}$$

Show that this mapping is linear, i.e., that

$$\lambda_1 \vec{r}_1 + \lambda_2 \vec{r}_2 \to \lambda_1 \vec{r}\,'_1 + \lambda_2 \vec{r}\,'_2 \tag{3}$$

under the mapping (here, λ_1 and λ_2 are constant coefficients). Then, argue that the mapping can alternatively be written as

$$\vec{r}\,' = \mathbb{R} \cdot \vec{r}\,, \tag{4}$$

where $\mathbb{R}$ is a 2×2 matrix whose entries need to be determined. Compare the entries of $\mathbb{R}$ with those of 2×2 rotation matrices and determine the rotation angle corresponding to the mapping (2). *Attempt to use scientific style in the writing of your solution.*

Task 2 (40 points)
Calculate the following residues:

$$\begin{aligned} R_1 &= \operatorname*{Res}_{z=0} \frac{1}{z^3\,\sin(z)}\,, \\ R_2 &= \operatorname*{Res}_{z=0} \frac{1}{z^4\,\exp(z)}\,, \\ R_3 &= \operatorname*{Res}_{z=1} \frac{1}{z\,(1-z)\,\exp(z)}\,, \\ R_4 &= \operatorname*{Res}_{z=1} \frac{1}{z\,(1-z)^2\,\exp(z)}\,. \end{aligned} \tag{5}$$

Some of these tasks have a little wit to them. Please read carefully. Not all residues are to be evaluated at $z = 0$. Also, remember that the Euler number is defined as $\exp(1) = \mathrm{e}^1 = \mathrm{e}$.

Task 3 (40 points)
Calculate the integral

$$I = \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{1}{(x^2+4)^2} \tag{6}$$

with the help of the Cauchy residue theorem. *For the geeks, unmarked: Compare to a numerical evaluation of the same integral, using your favorite numerical integration software.*

The tasks are due Thursday, 18–FEB–2021.