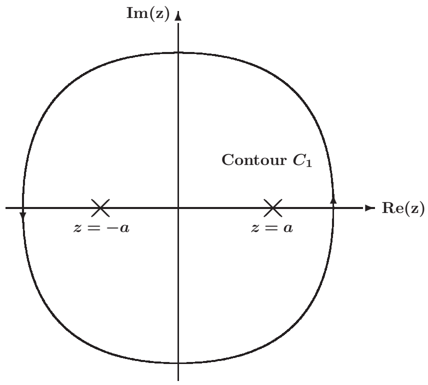
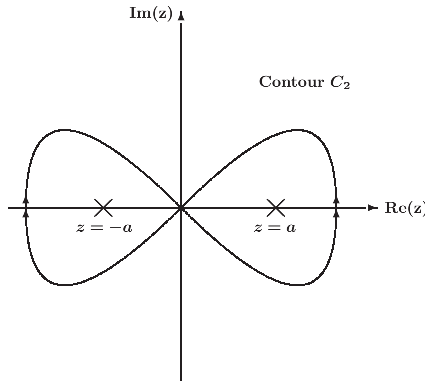
Physics/6403 Mathematical Physics I Dr. Ulrich Jentschura Spring Semester 2021
Missouri S & T (Rolla, Missouri) Exercise Sheet 5 Tuesday, 16–FEB–2021

Task 1 (40 points). Calculate the complex contour integrals I_1 and I_2, where

$$I_1 = \int_{C_1} \frac{1}{a^2 - z^2} \, \mathrm{d}z, \qquad I_2 = \int_{C_2} \frac{1}{a^2 - z^2} \, \mathrm{d}z \,, \tag{1}$$

where the contours C_1 and C_2 are given in the following figures (watch the little arrows).





Task 2 (40 points). It is argued in the lecture notes that the function

$$g_\eta(x) = \frac{1}{\pi} \frac{\eta}{x^2 + \eta^2} \tag{2}$$

approximates, for small η, a so-called Dirac δ function, so that, in the distributional sense ("under the integral sign") $\lim_{\eta \to 0^+} g_\eta(x) = \delta(x)$. Do a search on the internet, or, use your imagination, to find at least two other "representations of the Dirac–δ function, i.e., functions $g_\eta(x)$, which, in the limit $\eta \to 0$, have the properties

$$\int \delta(x) \, \mathrm{d}x = 0 \,, \qquad \int f(x) \, \delta(x) \, \mathrm{d}x = \lim_{\eta \to 0^+} \int f(x) \, g_\eta(x) \, \mathrm{d}x = f(0) \,, \tag{3}$$

for any test function $f = f(x)$. Here, x is a real variable.

Task 3 (40 points). Tabulate the following numerical values,

$$s_n = \sqrt{z_n} \,, \qquad z_n = 5.2 \, \exp\left(\mathrm{i} \frac{2\pi n}{360} \right) \,, \qquad n = 5, 15, 25, \dots, 175 \,, \tag{4}$$

i.e., the sequence of complex numbers obtained by taking the square root of a complex number of modulus $|z_n| = 5.2$, and with (in degrees) angles 5°, 15°, 25°, etc., up to 175°, in steps of 10 degrees. You may use your favorite computer algebra or computation system (*Mathematica*, MatLab, Julia, python, Jupyter, GiNAC, whatever. . .) in order to do the calculations. You should indicate the real and imaginary parts,

$$\mathrm{Re}\, s_n = \mathrm{Re}\, s_{5,15,\dots,175} \,, \qquad \mathrm{Im}\, s_n = \mathrm{Im}\, s_{5,15,\dots,175} \,, \tag{5}$$

separately, and numerically, to at least five or six decimals. Make a plot of the resulting s_n in the complex plane, i.e., indicate the position of the arguments z_n and the square roots s_n in the complex plane. Then, do the same for $s_n = \sqrt{z_n}$, where $z_n = 5.0 \exp(2\pi \mathrm{i} n/360)$, but this time, $n = -5, -15, -25, \dots, -175$, i.e., with arguments of the square root function which are in the lower half, as opposed to the upper half, of the complex plane. Calculate the quantities δ_1 and δ_2, where

$$\delta_1 = |z_{175} - z_{-175}| \,, \qquad \delta_2 = |s_{175} - s_{-175}| = \left| \sqrt{z_{175}} - \sqrt{z_{-175}} \right| \,. \tag{6}$$

Comment on the result. Why is δ_2 so much larger than δ_1?

The tasks are due, <u>by the very latest</u>, on Thursday, 25–FEB–2021 (for everyone).