Task 1 (unmarked, difficult, challenging). In some mathematics courses, we may learn that

$$\int \mathrm{d}x\, \frac{1}{x} = \ln(|x|)\,, \qquad x \in \mathbb{R}\,. \tag{1}$$

However, we may ask how the modulus $|x|$ made its way into the argument of the logarithm. Let us consider how the integral of the function

$$f(z) = \frac{1}{z} \tag{2}$$

should be formulated in the complex plane. I.e., we are looking for a function

$$g = g(z)\,, \qquad \frac{\mathrm{d}}{\mathrm{d}z} g(z) = \frac{1}{z}\,. \tag{3}$$

Let us investigate the question on the level of power series, where we assume that, in complex plane, within the circle of convergence, one has

$$\ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1}\, \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\,, \qquad |z| < 1\,. \tag{4}$$

Differentiating the power series with respect to z, show that one obtains

$$\frac{\mathrm{d}}{\mathrm{d}z} \ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1}\, z^{n-1} = \sum_{n=0}^{\infty} (-1)^n\, (-z)^n = 1 - z + z^2 - z^3 + z^4 + \cdots = \frac{1}{1+z}\,, \qquad |z| < 1\,. \tag{5}$$

Hence, since $\ln(1+z)$ is the integral of $1/(1+z)$, show that (under which additional assumptions!?) $\ln(z)$ (or its analytic continuation) is the integral of $1/z$.

The following questions immediately present themselves: How should the logarithm of a complex number be defined away from the real axis, i.e., in the entire complex plane, and how does a suitable definition of the complex logarithm affect the calculations away from the real axis? Also, how does the "conventional wisdom" summarized in Eq. (1) *hold up against checking against the analytic continuation into the complex plane?*

Task 2 (unmarked, difficult, challenging). Verify the Mittag–Leffler expansion of the cosecans,

$$\frac{1}{\sin(z)} = \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{z - n\pi} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2\,(-1)^k\, z}{z^2 - (k\pi)^2}\,, \qquad z \neq n\pi\,, \qquad n \in \mathbb{Z}\,, \tag{6}$$

for three complex example cases of your choice, by evaluating the complex sine function on the left-hand side, and comparing to a numerically calculated sum for the right-hand side.

The tasks are due at a date of your discretion.