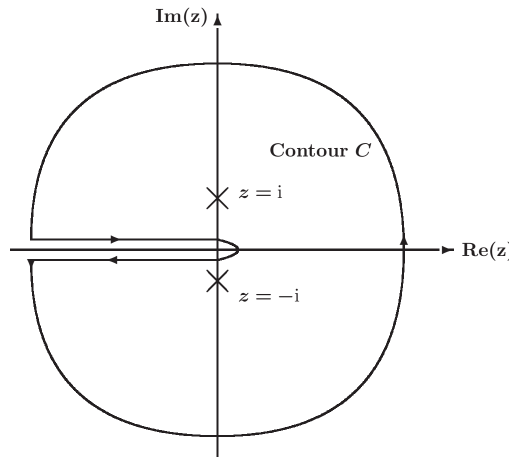

Task 1 (40 points). Consider a specific complex contour integral, namely

$$I = \int_C \mathrm{d}z \frac{\sqrt{z}}{z^2+1}\,, \tag{1}$$

where C goes infinitesimally above the real axis from $-\infty$ to zero, and it goes infinitesimally below the real axis from 0 to $-\infty$, and then, the closed contour is completed along a circle, along the mathematically positive direction, at a large radius $|z| = R \to \infty$.
A pertinent picture showing the contour is given on the right.

Show that the modulus of the integrand, for large $|z|$, is proportional to $1/|z|^{3/2}$. Show that the length of the contour, at large radius, is proportional to $2\pi R$, where R is the radius of the circle. Show that the contribution of the (almost complete) circle at radius R to the integral I therefore vanishes as the radius of the circle goes to infinity. Show that, taking the difference of the integrand immediately above and below the branch cut, and observing that the integrand vanishes sufficiently fast at large radius in the one obtains, by direct integration infinitesimally above and below the branch cut,

$$J = \int_C \mathrm{d}z \frac{\sqrt{z}}{z^2+1} = 2\int_0^\infty \mathrm{d}x \frac{2\mathrm{i}\sqrt{x}}{x^2+1} = \sqrt{2}\pi\mathrm{i}\,. \tag{2}$$

Explore an alternative way to evaluate the contour integral I. Show that the residues are given as follows,

$$R_1 = \operatorname*{Res}_{z=\mathrm{i}} \frac{\sqrt{z}}{z^2+1} = -\frac{1}{2}\exp\left(\frac{3}{4}\mathrm{i}\pi\right), \qquad R_2 = \operatorname*{Res}_{z=-\mathrm{i}} \frac{\sqrt{z}}{z^2+1} = \frac{1}{2}\exp\left(\frac{1}{4}\mathrm{i}\pi\right). \tag{3}$$

Show that the sum of the residues is

$$2\pi\mathrm{i}(R_1+R_2) = \sqrt{2}\pi\mathrm{i} = J\,. \tag{4}$$

Show that this result constitutes an alternative way to evaluate the integral J.

Show your work! Show all your work and all your intermediate steps! If your calculation is not well documented, significant reduction of marks will occur. If you use online tools, include a printout of your session!

Task 2 (40 points). Consider the Joukowsky mapping

$$z(\zeta) = \zeta + \frac{1}{\zeta}\,. \tag{5}$$

Consider a circle of radius 3/2 centered at the point $z_0 = 1/2$, i.e., the curve

$$\zeta = \zeta(t) = \frac{1}{2} + \frac{2}{3}\mathrm{e}^{\mathrm{i}t}\,, \qquad t \in (0, 2\pi)\,. \tag{6}$$

In the complex plane, draw the curve (show your work!)

$$w(t) = z(\zeta(t))\,, \qquad t \in (0, 2\pi)\,. \tag{7}$$

The tasks are due on Thursday, 04–MAR–2021.