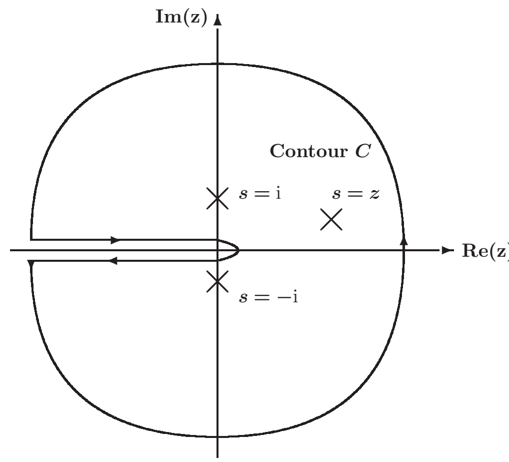
Task 1 (20 EXTRA points). Let us study dispersion relations by way of example. We consider the function

$$f(s) = \frac{\sqrt{s}}{s^2+1}\,, \qquad s \notin -\mathbb{R}_+ . \tag{1}$$

We study the integral

$$J(z) = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}s \, \frac{f(s)}{s-z}\,, \qquad z \notin \{-\mathrm{i}, \mathrm{i}, -\mathbb{R}_+\}\,, \tag{2}$$

where the contour C is given in the figure on the right. The integral $J(z)$ depends on the independent variable z, while s is the integration variable.



Justify (express your entire line of reasoning) the relation

$$J(z) = f(z) + \operatorname*{Res}_{s=-\mathrm{i}} \frac{f(s)}{s-z} + \operatorname*{Res}_{s=\mathrm{i}} \frac{f(s)}{s-z}\,. \tag{3}$$

Show the alternative representation

$$J(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{0} \mathrm{d}s \, \frac{f(s+\mathrm{i}\epsilon) - f(s-\mathrm{i}\epsilon)}{s-z}\,. \tag{4}$$

Comparing Eqs. (3) and (4), show that one obtains the dispersion relation

$$f(z) = -\operatorname*{Res}_{s=-\mathrm{i}} \frac{f(s)}{s-z} - \operatorname*{Res}_{s=\mathrm{i}} \frac{f(s)}{s-z} + \frac{1}{\pi} \int_{-\infty}^{0} \mathrm{d}s \, \frac{\mathrm{Im} f(s+\mathrm{i}\epsilon)}{s-z}\,. \tag{5}$$

Show that (show your work!)

$$F_1 \equiv -\operatorname*{Res}_{s=-\mathrm{i}} \frac{f(s)}{s-z} - \operatorname*{Res}_{s=\mathrm{i}} \frac{f(s)}{s-z} = \frac{1+z}{\sqrt{2}\,(1+z^2)}\,, \tag{6}$$

while

$$F_2 \equiv -\frac{1}{2\pi \mathrm{i}} \int_0^{\infty} \mathrm{d}t \, \frac{f(-t+\mathrm{i}\epsilon) - f(-t-\mathrm{i}\epsilon)}{-t-z} = -\frac{1}{2\sqrt{z} + \sqrt{2}\,(1+z)}\,. \tag{7}$$

Show that, adding F_1 and F_2, one obtains back $f(z)$,

$$f(z) = F_1 + F_2\,. \tag{8}$$

Task 2 (umarked). Justify the integral formula you know from undergraduate mathematics,

$$\int \mathrm{d}x \, \frac{1}{x} = \ln(|x|)\,, \tag{9}$$

by interpreting an integration region comprising negative x in terms of a complex contour, which is either displaced infinitesimally above, or infinitesimally below, the real axis.

The tasks are due on Thursday, 04–MAR–2021.