Task 1 (40 points). (a) Consider the velocity potential for incompressible flow around a cylinder,

$$\phi(x,y) = -u_\infty \left(x + \frac{a^2}{r^2} x \right) , \qquad r = \sqrt{x^2+y^2} \geq a \,. \tag{1}$$

Show that, in the asymptotic limit, for a point $(x,y) = x\,\hat{e}_x + y\,\hat{e}_y$ for from the origin, one has

$$\vec{\nabla}\phi(x,y) \to -u_\infty\,\hat{e}_x \,, \qquad \sqrt{x^2+y^2} \to \infty. \tag{2}$$

and interpret your result geometrically.

(b) Evaluate the gradient of the velocity potential,

$$\vec{u}(0,y) = \vec{\nabla}\phi(0,y) \,, \qquad |y| > a \,, \tag{3}$$

and argue that the maximum modulus of the velocity as the fluid moves around the cylinder in $2u_\infty$, i.e., twice the speed encountered at infinity.

Hint: You might obtain the vector-valued result

$$\vec{u}(0,y) = -u_\infty \left(1 + \frac{a^2}{y^2} \right) \hat{e}_x \,, \tag{4}$$

but this needs to be checked. Show all your work!

Task 2 (40 points). Show, by an explicit calculation, using the Cartesian representation of the Laplacian, that

$$\vec{\nabla}^2\phi(x,y) = 0 \,. \tag{5}$$

Interpret your result in terms of the continuity equation and the incompressibility of the fluid.

Hint: This calculation is more involved and more difficult than it seems at first. Show all intermediate steps. Show *all* intermediate steps and do *not* refer to any general results. Your calculation must be reproducible and explicit.

Task 3 (40 points). Consider the complex potential

$$w = w(z) = -u_\infty \left(z + \frac{a^2}{z} \right) , \qquad z = x + \mathrm{i}y \tag{6}$$

Decompose w into real and imaginary parts, i.e., calculate the velocity potential ϕ and the stream functionψ, in

$$w = \phi + \mathrm{i}\,\psi \,. \tag{7}$$

Show that

$$\phi = -\,u_\infty \left(x + \frac{a^2}{r^2} x \right) , \tag{8}$$

$$\psi = -\,u_\infty \left(y - \frac{a^2}{r^2} y \right) . \tag{9}$$

Show that the stream function $\psi(x,y)$ assumes a constant value (which one?) along the line $y=0$ (trivial question) and also along the border of the circle, at $r^2 = a^2$, and interpret your result physically (to the extent possible).

The tasks are due on Thursday, 11–MAR–2021.