Task 1 (50 points). We had previously discussed the Cauchy–Riemann equations for a complex function

$$f(z) = f_1(z) + \mathrm{i} f_2(z) = f_1(x,y) + \mathrm{i} f_2(x,y)\,, \tag{1}$$

where $f_1(x,y)$ and $f_2(x,y)$ are real rather than complex. For the physics of airfoils, we had considered a general complex potential $w(z)$, a velocity potential $\phi(x,y)$, and a stream function $\psi(x,y)$,

$$w(z) = \phi(z) + \mathrm{i}\psi(z) = \phi(x,y) + \mathrm{i}\psi(x,y)\,. \tag{2}$$

Convince yourself that the relations obtained previously for f_1 and f_2 are consistent with those that we now obtain for ϕ and ψ, i.e., that we just "changed the notation", in order to be consistent with the literature on fluid flow, where the complex potential w is widely used.

Show your work and all intermediate steps!

Give a reference when you use equations from the lecture notes!

Task 2 (50 points). Diffential ratios, which give rise to derivatives, work by investigating the change in the value of a function as you change the argument of the functions. Consider the change of a complex function $f(z)$, according to

$$f(z) \to f(z + \delta z_1)\,, \tag{3}$$

under an infinitesimal change along the x axis,

$$z = x + \mathrm{i}y \to z + \delta z_1 = x + \mathrm{i}y + \delta x\,, \qquad \delta z_1 = \delta x\,. \tag{4}$$

Alternatively, consider the change of a complex function $f(z)$, according to

$$f(z) \to f(z + \delta z_2)\,, \tag{5}$$

under an infinitesimal change along the y axis,

$$z = x + \mathrm{i}y \to z + \delta z_2 = x + \mathrm{i}y + \mathrm{i}\delta y\,, \qquad \delta z_2 = \mathrm{i}\delta y\,. \tag{6}$$

Write equations which relate the complex derivative $\mathrm{d}f(z)/\mathrm{d}z$ in terms of δz_1, δz_2, δx, and δy, and limiting processes. Also, *derive* a form of the Cauchy–Riemann equations which clarifies that the value of the complex derivative does not depend on the direction in which the complex derivative is taken, i.e., that you can define the complex derivative either in terms of infinitesimal changes of the complex argument along the real or imaginary axes, and still obtain consistent results.

Show your work and all intermediate steps!

Give a reference when you use equations from the lecture notes!

Please turn over!

Task 3 (100 extra points). Consider the Γ function, which, for $\operatorname{Re} z > 0$ is given by

$$\Gamma(z) = \int_0^\infty \exp(-t)\, t^{z-1}\, \mathrm{d}t = \int_0^1 \left[\ln(1/t)\right]^{z-1}\, \mathrm{d}t\,. \tag{7}$$

(a) Show the equivalence of both integral representations!

(b) Split the integration interval as follows:

$$\Gamma(z) = \int_0^\infty \exp(-t)\, t^{z-1}\, \mathrm{d}t = P(z) + Q(z)\,, \tag{8}$$

$$P(z) = \int_0^1 \exp(-t)\, t^{z-1}\, \mathrm{d}t\,, \tag{9}$$

$$Q(z) = \int_1^\infty \exp(-t)\, t^{z-1}\, \mathrm{d}t\,. \tag{10}$$

By expanding the exponential in the integrand of $P(z)$ in a power series and integrating term by term (how? your task…), show that $\Gamma(z)$ has simple poles for $z = -n$, with $n \in \mathbb{R}_0$, and calculate their residues!

Show that $Q(z)$ is an *entire function*. What is an *entire function*? Your task to find out…

From now on, but not above, we consider cases with $z > 0$*, i.e.,* $z \in \mathbb{R}$*, and* $z = x$*, or* $z = y$*, with* $x \in \mathbb{R}$*, and* $y \in \mathbb{R}$*.*

(c) Look up *Stirling's approximation* to $\Gamma(x)$ for large $x > 0$, and consider how to derive this approximation.

(d) Define the beta function, denoted as

$$B(x, y) = \int_0^1 t^{x-1}\, (1-t)^{y-1}\, \mathrm{d}t\,. \tag{11}$$

Show that (and this is really difficult)

$$B(x, y) = \frac{\Gamma(x)\, \Gamma(y)}{\Gamma(x+y)}\,. \tag{12}$$

Task 3 is the subject of a project session on Thursday, 01–APR–2021, where you can discuss the intermediate steps necessary for the complex derivations required for task # 3.

The tasks are due on Tuesday, 06–APR–2021.