

Task 1 (40 points). Consider spherical coordinates, with the position vector \vec{r} being written as follows,

$$\vec{r} = r \sin \theta \cos \varphi \hat{e}_x + r \sin \theta \sin \varphi \hat{e}_y + r \cos \theta \hat{e}_z. \quad (1)$$

Calculate the physical (unit) basis vectors

$$\hat{e}_r = \left(\left| \frac{\partial \vec{r}}{\partial r} \right| \right)^{-1} \frac{\partial \vec{r}}{\partial r}, \quad \hat{e}_\theta = \left(\left| \frac{\partial \vec{r}}{\partial \theta} \right| \right)^{-1} \frac{\partial \vec{r}}{\partial \theta}, \quad \hat{e}_\varphi = \left(\left| \frac{\partial \vec{r}}{\partial \varphi} \right| \right)^{-1} \frac{\partial \vec{r}}{\partial \varphi}. \quad (2)$$

Now consider a time-dependent position vector,

$$\vec{r}(t) = r(t) \sin \theta(t) \cos \varphi(t) \hat{e}_x + r(t) \sin \theta(t) \sin \varphi(t) \hat{e}_y + r(t) \cos \theta(t) \hat{e}_z. \quad (3)$$

Show that, with the common notation $\dot{f}(t) \equiv df(t)/dt$, one has

$$\dot{\vec{r}}(t) = \dot{r}(t) \hat{e}_r + \dot{\theta}(t) r(t) \hat{e}_\theta + \dot{\varphi}(t) r(t) \sin \theta(t) \hat{e}_\varphi. \quad (4)$$

Task 2 (40 points). Write the general transformation laws for the contravariant and covariant components of a vector. Show that, if one parameterizes space by three *general* coordinates x^1 , x^2 , and x^3 , then the vectors

$$\vec{e}_i \equiv \frac{\partial \vec{r}}{\partial x^i}, \quad i = 1, 2, 3, \quad (5)$$

transform with the covariant transformation law. Also, show that the components of the gradient vector,

$$\left(\vec{\nabla} f \right)_i \equiv \frac{\partial f}{\partial x^i}, \quad i = 1, 2, 3, \quad (6)$$

transform as covariant components.

For the spherical coordinates given above ($x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$), calculate the covariant vectors

$$\vec{e}_1 \equiv \vec{e}_r = \frac{\partial \vec{r}}{\partial r}, \quad \vec{e}_2 \equiv \vec{e}_\theta = \frac{\partial \vec{r}}{\partial \theta}, \quad \vec{e}_3 \equiv \vec{e}_\varphi = \frac{\partial \vec{r}}{\partial \varphi}. \quad (7)$$

Show that you can express the velocity in terms of contravariant vector components \dot{x}^i and covariant basis vector \vec{e}_i as

$$\dot{\vec{r}} = \dot{r} \vec{e}_r + \dot{\theta} \vec{e}_\theta + \dot{\varphi} \vec{e}_\varphi = \sum_{i=1}^3 \dot{x}^i \vec{e}_i, \quad d\vec{r} = dr \vec{e}_r + \vec{e}_\theta d\theta + \vec{e}_\varphi d\varphi. \quad (8)$$

Show that the latter form is obtained by multiplying the former by a infinitesimal time interval dt .

Watch the difference between the \hat{e}_i and the \vec{e}_i !!!

Task 3 (40 points). Calculate the modulus of the displacement

$$d\vec{r}^2 = (dr \vec{e}_r + \vec{e}_\theta d\theta + \vec{e}_\varphi d\varphi)^2 = g_{ij} dx^i dx^j, \quad (9)$$

with $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, and identify the components of the metric tensor

$$g_{ij} = ?, \quad i, j = 1, 2, 3. \quad (10)$$

Show that, in the general case, the components of g_{ij} transform as the components of a second-rank, covariant tensor. Show also that, if the A_i are covariant vector components, then the components A^i , defined as

$$A^i = \sum_{j=1}^3 g^{ij} A_j, \quad (11)$$

transform as contravariant components. **For 20 extra points, calculate the contravariant components of the contravariant metric tensor g^{ij} for spherical coordinates.**