Task 1 (40 points). Start from the invariance of the full physical vector under coordinate transformations,

$$\vec{A} = A^i \, \vec{e}_i = \widetilde{A}^i \, \widetilde{\vec{e}}_i \,, \tag{1}$$

and using the quotient rule as discussed in the lecture, show that the covariant derivative

$$\nabla_j A^i \equiv \frac{\partial A^i}{\partial x^j} + \Gamma^i_{jk} \, A^k \,, \qquad \Gamma^i_{jk} = \frac{\partial \vec{e}_j}{\partial x^k} \cdot \vec{e}^{\,i} \tag{2}$$

is a mixed convariant-contravarinat tensor of second rank. Furthermore, show the following identity for the Christoffel symbols of the second kind,

$$\Gamma^i_{jk} = \Gamma^i_{kj} \,. \tag{3}$$

Task 2 (40 points). Show that the Christoffel symbols of the first kind,

$$\Gamma_{ijk} = \frac{\partial \vec{e}_i}{\partial x^j} \cdot \vec{e}_k \,, \tag{4}$$

are symmetric in the first two indices, i.e., that

$$\Gamma_{ijk} = \Gamma_{jik} \,. \tag{5}$$

Then, starting from the definition given in Eq. (4), show that the Christoffel symbols of the first kind can be written as

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right) \,. \tag{6}$$

Task 3 (40 points). Show that, for spherical coordinates r, θ, and φ, with $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, defined (in the usual way) so that

$$\vec{r} = r \, \sin\theta \, \cos\varphi \, \hat{e}_x + r \, \sin\theta \, \sin\varphi \, \hat{e}_y + r \, \cos\theta \, \hat{e}_z \,, \tag{7}$$

one has the following Christoffel symbols of the first kind,

$$\Gamma_{221} = -r = -\Gamma_{122} = -\Gamma_{212} \,, \tag{8a}$$

$$\Gamma_{331} = -r \sin^2\theta = -\Gamma_{133} = -\Gamma_{313} \,, \tag{8b}$$

$$\Gamma_{332} = -\tfrac{1}{2} r^2 \sin(2\theta) = -\Gamma_{233} = -\Gamma_{323} \,. \tag{8c}$$

Furthermore, show that the Christoffel symbols of the second kind are given as follows,

$$\Gamma^1_{22} = -r \,, \tag{9a}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = 1/r \,, \tag{9b}$$

$$\Gamma^2_{33} = -\tfrac{1}{2} \sin(2\theta) \,, \tag{9c}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \,. \tag{9d}$$

The tasks are due on Thursday, 22–APR–2021.