Task 1 (90 points). Consider the following complex function,

$$f(z) = z\,\cos(z) = f(x + \mathrm{i}y) = (x + \mathrm{i}y)\,\cos(x + \mathrm{i}y)\,, \qquad x = \mathrm{Re}\,z\,, \qquad y = \mathrm{Im}\,z\,, \tag{1}$$

(a) Calculate the following function values numerically:

$$f(1), \qquad f(-1), \qquad f(1+\mathrm{i}), \qquad f(4.3-2.3\mathrm{i}), \qquad f(4.3+2.3\mathrm{i})\,. \tag{2}$$

Show your work and all intermediate steps. If you use a calculator, specify at which point it has been used, and how!

Solution:

The following results are obtained by direct use of a calculator,

$$f(1) = 1 \times \cos(1) = \cos(1) = 0.540302\,, \tag{3}$$

$$f(-1) = (-1)\,\times \cos(-1) = -\cos(-1) = -0.540302\,. \tag{4}$$

If one does not have a calculator, then one remembers that

$$\cos(1) = \cos(1\,\mathrm{rad}) = \cos\left(\frac{360^\circ}{2\pi}\right) \approx \cos(60^\circ) = \frac{1}{2} \tag{5}$$

This gives at least one significant figure in the result. If one does not remember that $\cos(60^\circ) = \frac{1}{2}$, one can immediately rederive the result by considering a triangle with equal side lengths, each inner angle being 60°, and dividing one of the inner angles by two, and drawing a line which intersects the opposite side of the triangle in the middle. By the Pythagoras theorem,

$$\sin(30^\circ) = \sqrt{1 - 1/4} = \sqrt{3/4} = \frac{\sqrt{3}}{2} \approx \frac{1.7}{2} \approx 0.85\,. \tag{6}$$

For the next task, one has to evaluate

$$\cos(x + \mathrm{i}y) = \frac{\mathrm{e}^{\mathrm{i}(x+\mathrm{i}y)} + \mathrm{e}^{-\mathrm{i}(x+\mathrm{i}y)}}{2} \tag{7}$$

$$= \frac{1}{2}\,\mathrm{e}^{-y}\,[\cos(x) + \mathrm{i}\sin(x)] + \frac{1}{2}\mathrm{e}^{y}\,[\cos(x) - \mathrm{i}\sin(x)] \tag{8}$$

So,

$$\begin{aligned}\cos(1+\mathrm{i}) &= \frac{1}{2} \times \exp(-1) \times [\cos(1) + \mathrm{i}\sin(1)] + \frac{1}{2}\exp(1)\,[\cos(1) - \mathrm{i}\sin(1)] \\ &\approx \frac{1}{2} \times \frac{1}{2.7} \times [0.5 + \mathrm{i}\,0.85] + \frac{1}{2} \times 2.7 \times [0.5 - \mathrm{i}\,0.85] \\ &= 0.67 + 1.14\mathrm{i} + 0.09 - 0.15\,i = 0.77 - 0.99\,\mathrm{i}\,.\end{aligned} \tag{9}$$

The exact value is

$$\cos(1+\mathrm{i}) = 0.83373 - 0.98889\,\mathrm{i}\,, \tag{10}$$

and the correspondence is good enough. Multiplying, one obtains

$$\begin{aligned}f(1+\mathrm{i}) &= (1+\mathrm{i})\cos(1+\mathrm{i}) = (1+\mathrm{i})(0.83373 - 0.98889\,\mathrm{i}) \\ &= 0.83373 + 0.98889 + (0.83373 - 0.98889)\,\mathrm{i} = 1.82262 - 0.15516\,\mathrm{i}\,.\end{aligned} \tag{11}$$

Likewise, one obtains

$$f(4.3 - 2.3\mathrm{i}) = (4.3 - 2.3\mathrm{i})\,\cos(4.3 - 2.3\mathrm{i}) = -19.0844 - 14.8057\mathrm{i}\,, \tag{12}$$

$$f(4.3 + 2.3\mathrm{i}) = (4.3 + 2.3\mathrm{i})\,\cos(4.3 + 2.3\mathrm{i}) = -19.0844 + 14.8057\mathrm{i}\,. \tag{13}$$

The latter two results are complex conjugates of each other.

(b) Calculate the complex derivatives, and partial derivatives:

$$P_1 \equiv \frac{\mathrm{d}f(z)}{\mathrm{d}z}\,, \qquad P_2 \equiv \frac{\partial f(x+\mathrm{i}y)}{\partial x}\,, \qquad P_3 \equiv -\mathrm{i}\frac{\partial f(x+\mathrm{i}y)}{\partial y}\,, \tag{14}$$

where f is given in Eq. (1). If you should find that $P_1 = P_2 = P_3$, explain why!

Solution:

By applying formulas for the derivative of trigonometric functions, which hold in the complex domain, and using the product rule, one obtains

$$P_1 = \frac{\mathrm{d}f(z)}{\mathrm{d}z} = \cos z - z\sin z \tag{15}$$

$$P_2 = \frac{\partial f(x+\mathrm{i}y)}{\partial x} = \cos(x+\mathrm{i}y) - (x+\mathrm{i}y)\sin(x+\mathrm{i}y) \tag{16}$$

$$= \cos z - z\sin z\,, \tag{17}$$

$$P_3 = -\mathrm{i}\frac{\partial f(x+\mathrm{i}y)}{\partial y} = -\mathrm{i}\left[\mathrm{i}\cos(x+\mathrm{i}y) - \mathrm{i}(x+\mathrm{i}y)\sin(x+\mathrm{i}y)\right] \tag{18}$$

$$= \cos(x+iy) - (x+\mathrm{i}y)\sin(x+\mathrm{i}y) \tag{19}$$

$$= \cos(z) - z\sin(z)\,. \tag{20}$$

In the lecture, we had already shown that the value of a complex derivative of a function does not depend on the direction in which it is taken in the complex plane. The complex derivative is taken as

$$\frac{ddf(z)}{\mathrm{d}z} = \lim_{\delta z\to 0}\frac{f(z+\delta z) - f(z)}{\delta z}\,, \tag{21}$$

and it does not matter in which direction δz is pointing. P_1 is obtained for general δz, P_2 is obtained by setting $\delta z = \delta x$ (variation of the complex argument along the real axis), and P_3 is obtained by setting $\delta z = \mathrm{i}\,\delta y$ (variation of the complex argument along the imaginary axis).

(c) Write the Cauchy–Riemann equations in at least one of the alternative forms discussed in the lecture. Show that the function $f(z) = z\,\cos(z)$ fulfills the Cauchy–Riemann partial differential equations! Show your work and all intermediate steps!

Solution: If we define the vector

$$\vec{F}^*(z) = \hat{e}_x f_1(z) - \hat{e}_y f_2(z)\,, \tag{22}$$

we can write the Cauchy–Riemann equations

$$\vec{\nabla}\cdot\vec{F}^*(z) = \left(\hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial z}\right)\cdot\left(\hat{e}_x f_1(z) - \hat{e}_y f_2(z)\right) = \frac{\partial f_1(x+\mathrm{i}y)}{\partial x} - \frac{\partial f_2(x+\mathrm{i}y)}{\partial z} = 0 \tag{23}$$

We can separate $f(z)$ into its real and imaginary parts by taking Eq. (7) one step further,

$$\cos(x+\mathrm{i}y) = \frac{\mathrm{e}^{\mathrm{i}(x+\mathrm{i}y)} + \mathrm{e}^{-\mathrm{i}(x+\mathrm{i}y)}}{2} \tag{24}$$

$$= \frac{1}{2}\,\mathrm{e}^{-y}\left[\cos(x) + \mathrm{i}\sin(x)\right] + \frac{1}{2}\mathrm{e}^{y}\left[\cos(x) - \mathrm{i}\sin(x)\right] \tag{25}$$

$$= \frac{1}{2}\,\left(\mathrm{e}^{y} + \mathrm{e}^{-y}\right)\,\cos(x) - \frac{\mathrm{i}}{2}\,\left(\mathrm{e}^{y} - \mathrm{e}^{-y}\right)\,\sin(x) \tag{26}$$

$$= \cosh(y)\,\cos(x) - \mathrm{i}\,\sinh(y)\,\sin(x) \tag{27}$$

So,

$$f(z) = (x+\mathrm{i}y)\,\cos\left(x+\mathrm{i}y\right) \tag{28}$$

$$= (x+\mathrm{i}y)\left[\cosh(y)\,\cos(x) - \mathrm{i}\,\sinh(y)\,\sin(x)\right] \tag{29}$$

so

$$f_1(x,y) = x\cos(x)\cosh(y) + y\sin(x)\sinh(y)\,, \tag{30}$$

(31)

$$f_2(x,y) = y\cos(x)\cosh(y) - x\sin(x)\sinh(y)\,. \tag{32}$$

Thus using Eq. (23) we can write,

$$\frac{\partial f_1(x,y)}{\partial x} = -\,x\sin(x)\cosh(y) + \cos(x)\cosh(y) + y\cos(x)\sinh(y)\,, \tag{33}$$

$$\frac{\partial f_2(x,y)}{\partial y} = y\cos(x)\sinh(y) + \cos(x)\cosh(y) - x\sin(x)\cosh(y)\,. \tag{34}$$

and thus the Cauchy-Riemann equations are fulfilled

$$\frac{\partial f_1(x,y)}{\partial x} = \frac{\partial f_2(x,y)}{\partial y}\,, \qquad \vec{\nabla}\cdot\vec{F}^*(z) = 0\,. \tag{35}$$

Task 2 (90 points + 20 extra points)). Start from the the velocity potential $w = w(z)$ for laminar incompressible flow around a cylinder (u_∞ is the velocity modulus far away from the cylinder),

$$w(z) = -u_\infty\left(z + \frac{a^2}{z}\right)\,, \qquad z = x + \mathrm{i}y\,, \qquad u_\infty \in \mathbf{R}\,, \quad \mathbf{x} \in \mathbf{R}\,, \quad \mathbf{y} \in \mathbf{R}\,. \tag{36}$$

(a) Show that the real part of $w(z)$ is given by

$$\mathrm{Re}\,w(z) = \phi(x,y) = -u_\infty\left(x + \frac{a^2\,x}{x^2+y^2}\right)\,. \tag{37}$$

(Solution:) An elementary calculation shows that

$$\begin{aligned} w(z) &= -\,u_\infty\left(z + \frac{a^2}{z}\right) \\ &= -\,u_\infty\left(x + \mathrm{i}y + \frac{a^2}{x+\mathrm{i}y}\right) \\ &= -\,u_\infty\left(x + \mathrm{i}y + \frac{a^2(x-\mathrm{i}y)}{(x+\mathrm{i}y)(x-\mathrm{i}y)}\right) \\ &= -\,u_\infty\left(x + \mathrm{i}y + \frac{a^2(x-\mathrm{i}y)}{(x^2+y^2)}\right) \\ &= -\,u_\infty\left(x + \frac{a^2x}{(x^2+y^2)}\right) - \mathrm{i}u_\infty\left(y - \frac{a^2y}{(x^2+y^2)}\right)\,. \end{aligned} \tag{38}$$

Thus, one has the relation

$$\mathrm{Re}\,w(z) = \phi(x,y) = -u_\infty\left(x + \frac{a^2x}{(x^2+y^2)}\right)\,, \tag{39}$$

which needed to be shown.

(b) Calculate the partial derivatives

$$u_x = \frac{\partial\phi}{\partial x}\,, \qquad u_y = \frac{\partial\phi}{\partial y}\,. \tag{40}$$

Solution: An application of elementary rules of differentiation (product, chain and sum rule) leads to the result

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= -\,u_\infty\left[\frac{a^2}{x^2+y^2} + a^2\,x\left(-\frac{1}{x^2+y^2}\right)^2(2x) + 1\right] \\ &= -\,u_\infty\left(\frac{a^2}{x^2+y^2} - \frac{2a^2x^2}{(x^2+y^2)^2} + 1\right) \\ &= -\,u_\infty\left(\frac{a^2\left(y^2-x^2\right)}{(x^2+y^2)^2} + 1\right)\,. \end{aligned} \tag{41}$$

The y derivative leads to

$$\frac{\partial\phi}{\partial y} = \frac{\partial}{\partial y}\left[-u_\infty\left(x + \frac{a^2 x}{(x^2+y^2)}\right)\right] = (-u_\infty)\, a^2 x \left(-\frac{1}{(x^2+y^2)^2}\right)(2y) = 2u_\infty \frac{a^2 xy}{(x^2+y^2)^2}\,. \tag{42}$$

(c) Calculate the complex derivative $\mathrm{d}w(z)/\mathrm{d}z$ and verify that

$$u_x = \mathrm{Re}\,\frac{\mathrm{d}w(z)}{\mathrm{d}z}\,, \qquad u_y = -\mathrm{Im}\,\frac{\mathrm{d}w(z)}{\mathrm{d}z}\,. \tag{43}$$

(Solution:) An elementary differentiation leads to

$$\begin{aligned}\frac{\mathrm{d}w(z)}{\mathrm{d}z} &= -\,u_\infty\left(1 - \frac{a^2}{z^2}\right)\\ &= -\,u_\infty\left(1 - \frac{a^2}{(x+\mathrm{i}y)^2}\right)\\ &= -\,u_\infty\left(1 - \frac{a^2(x-\mathrm{i}y)^2}{(x+\mathrm{i}y)^2(x-\mathrm{i}y)^2}\right)\\ &= -\,u_\infty\left(1 - \frac{a^2(x-\mathrm{i}y)^2}{(x^2+y^2)^2}\right)\\ &= -\,u_\infty\left(1 + \frac{a^2(y^2-x^2)}{(x^2+y^2)^2}\right) - \mathrm{i}\,2u_\infty \frac{a^2 xy}{(x^2+y^2)^2}\,.\end{aligned} \tag{44}$$

Thus, by comparing to solution in part b, we clearly find that

$$\begin{aligned}u_x &= \mathrm{Re}\frac{\mathrm{d}w(z)}{\mathrm{d}z}\\ u_y &= -\,\mathrm{Im}\frac{\mathrm{d}w(z)}{\mathrm{d}z}\,.\end{aligned} \tag{45}$$

(d) (20 extra points) Show that there exists a connection (which one?) between the two quantities Q_1 and Q_2, which are defined as follows,

$$Q_1 \equiv \frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2}\,, \qquad Q_2 \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi(x,y) = \vec{\nabla}^2\phi(x,y)\,. \tag{46}$$

Show, based on general or specific arguments, that Q_2 vanishes except at singular points.

Solution: The easiest way to see the connection is to refer to the equality of the expressions P_1, P_2 and P_3 above (part 1b).

From

$$\frac{\partial w(z)}{\partial z} = \frac{\partial w(x+\mathrm{i}y)}{\partial x} = -\mathrm{i}\frac{\partial w(x+\mathrm{i}y)}{\partial y}\,, \tag{47}$$

one obtains

$$Q_1 = \frac{\partial^2 w(z)}{\partial z^2} = \frac{\partial^2 w(x+\mathrm{i}y)}{\partial x^2} = (-\mathrm{i})^2\frac{\partial^2 w(x+\mathrm{i}y)}{\partial y^2}\,, \tag{48}$$

and hence

$$\frac{\partial^2 w(x+\mathrm{i}y)}{\partial x^2} + \frac{\partial^2 w(x+\mathrm{i}y)}{\partial y^2} = 0\,. \tag{49}$$

Hence, applying this equation to both the real as well as the imaginary parts of w (with $\phi = \mathrm{Re}\,w$ and $\psi = \mathrm{Im}\,w$), one obtains

$$Q_2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathrm{Re}\,w = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathrm{Re}\,\phi = 0 \tag{50}$$

and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathrm{Re}\,w = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\mathrm{Re}\,\psi = 0\,. \tag{51}$$