Task 1 (20 points)
Show that for a scalar field $\Psi = \Psi(\vec{r})$,

$$\oint_{\partial A} \Psi(\vec{s})\, \mathrm{d}\vec{s} = \int_A \mathrm{d}\vec{A} \times \vec{\nabla}\Psi(\vec{r}) = \int_A \hat{n} \times \vec{\nabla}\Psi(\vec{r})\, \mathrm{d}A\,. \tag{1}$$

The latter two expressions are equal by definition, the first equality is to be shown. The method of proof is to apply Stokes's theorem to a vector field $\vec{V}(\vec{s}) = \vec{b}\,\Psi(\vec{s})$, where $\vec{b}$ is an arbitary constant vector. Please use this method.

Task 2 (20 points)
Interpret the equation

$$\Phi(\vec{r}) = -\frac{1}{\epsilon_0} \int \mathrm{d}^3 r'\, g(\vec{r} - \vec{r}\,')\, \rho(\vec{r}\,')\,, \qquad g(\vec{r} - \vec{r}\,') = -\frac{1}{4\pi|\vec{r} - \vec{r}\,'|}\,, \tag{2}$$

with symbols defined as in the lecture (I hope you took notes), in terms of the Green function formalism. Write an essay with full English sentences, using correct grammar and scientific expressions. What is the "purpose" of a Green function in general?

Task 3 (30 points)
Define

$$||\vec{r}|| = \sqrt{x^2 + y^2 + z^2}\,, \qquad x_1 = x\,, \quad x_2 = y\,, \quad x_3 = z\,. \tag{3}$$

Show that for all $i, j = 1, 2, 3$, one has the relation

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{1}{||\vec{r}||} = \Theta(||\vec{r}||) \left\{ \frac{3 x_i x_j}{||\vec{r}||^5} - \frac{\delta_{ij}}{||\vec{r}||^3} \right\} - \frac{4\pi}{3} \delta_{ij}\, \delta^{(3)}(\vec{r})\,. \tag{4}$$

Here, $\Theta(x)$ is the Heaviside step function, defined so that $\Theta(x) = 0$ for $x \leq 0$, and $\Theta(x) = 1$ for $x > 0$. Why is this result compatible with the defining equation for the Green function of the Poisson equation in three dimensions? Write a little essay on what happens when you let $i = j$ and sum over i from 1 to 3.

Task 4 (30 points)
Derive the wave equations

$$\left(\vec{\nabla}^{\,2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = \vec{0}\,, \qquad \left(\vec{\nabla}^{\,2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{B} = \vec{0}\,, \tag{5}$$

for the electric and magnetic fields in a source-free region, $\rho = 0$ and $\vec{J} = 0$, by working in the SI mksA unit system in all intermediate steps.

Task 5 (20 points)
Take the divergence of both sides of the Ampere–Maxwell law,

$$\vec{\nabla} \times \vec{B}(\vec{r}, t) = \mu_0\, \vec{J}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E}(\vec{r}, t)\,, \tag{6}$$

and show that you obtain the time derivative of Gauss's law.

The tasks are due Thursday, 30-SEP-2021. Have fun doing the problems!