Task 1 (40 points)
Verify the relation

$$\int Y^*_{\ell' \, m'}\,(\theta,\phi)\; Y_{\ell m}\,(\theta,\phi)\; \mathrm{d}\Omega = \int_0^{2\pi} \int_0^{\pi} Y^*_{\ell' \, m'}\,(\theta,\phi)\; Y_{\ell m}\,(\theta,\phi)\; \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi = \delta_{\ell\ell'}\, \delta_{mm'}\,, \tag{1}$$

by an explicit calculation for all spherical harmonics with $\ell = 0, 1$.

Task 2 (40 points)
Verify the relation ($\ell = 2$)

$$\int Y^*_{2m}\,(\theta,\phi)\; Y_{2m}\,(\theta,\phi)\; \mathrm{d}\Omega = 1 \qquad \text{(no summation over } m!) \tag{2}$$

by an explicit calculation for all spherical harmonics with $\ell = 2$, and $m = -2, -1, 0, 1, 2$.

Task 3 (100 points)
In the lecture, we had derived the relation

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\, \frac{r_<^{\ell}}{r_>^{\ell+1}}\, Y_{\ell m}\,(\theta,\varphi)\; Y^*_{\ell m}\,(\theta',\varphi')\,, \tag{3}$$

as the multipole expansion of the Green function of the three-dimensional Possion equation.

Write an essay about the multipole expansion, and consider at least two, slightly different, ways of deriving Eq. (3), as outlined in the lecture notes. Hint: consider an appropriate angular decomposition, and note that the Green function of the Poisson equation fulfills the homogeneous equation except in the immediate vicinity of $\vec{r} = \vec{r}'$.

Task 3 (100 EXTRA!!! points)
Consider the parameters

$$\vec{r} = \vec{r}_1 = 5\,\hat{\mathrm{e}}_x + 3\hat{\mathrm{e}}_y + 2\hat{\mathrm{e}}_z\,, \qquad \vec{r}' = \vec{r}_2 = 0.1\,\hat{\mathrm{e}}_x + 0.2\hat{\mathrm{e}}_y + 0.4\hat{\mathrm{e}}_z\,. \tag{4}$$

Define the terms

$$\begin{aligned} T_\ell &= \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\, \frac{r_<^{\ell}}{r_>^{\ell+1}}\, Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta',\varphi') \\ &= \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\, \frac{r_2^{\ell}}{r_1^{\ell+1}}\, Y_{\ell m}(\theta_1,\varphi_1)\, Y^*_{\ell m}(\theta_2,\varphi_2)\,, \end{aligned} \tag{5}$$

where the second line is just a trivial specialization of the first, to the case $\vec{r} = \vec{r}_1$ and $\vec{r}' = \vec{r}_2$, and we anticipate that $r_2 = r_<$, and $r_1 = r_>$ (why?). Write a computer symbolic program which calculates, explicitly and numerically,

$$T_{\ell=0,1,2,3,4,5,6,7,8}\,, \qquad \text{(nine terms)} \tag{6}$$

Show that the sum converges fast, and that the result

$$\frac{1}{|\vec{r} - \vec{r}'|} \approx \sum_{\ell=0}^{8} T_\ell \tag{7}$$

holds to at least six decimal figures.

The tasks are due Thursday, 28–OCT–2021.