Task 1 (100 EXTRA!!! points)
Consider the parameters

$$\vec{r} = \vec{r}_1 = 5.5\,\hat{\mathrm{e}}_x + 3.3\hat{\mathrm{e}}_y + 2.3\hat{\mathrm{e}}_z\,, \qquad \vec{r}' = \vec{r}_2 = 5.1\,\hat{\mathrm{e}}_x + 3.1\hat{\mathrm{e}}_y + 2.2\hat{\mathrm{e}}_z\,. \tag{1}$$

Define the terms

$$\begin{aligned} T_\ell &= \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,Y_{\ell m}(\theta,\varphi)\,Y^*_{\ell m}(\theta',\varphi') \\ &= \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\,\frac{r_2^\ell}{r_1^{\ell+1}}\,Y_{\ell m}(\theta_1,\varphi_1)\,Y^*_{\ell m}(\theta_2,\varphi_2)\,, \end{aligned} \tag{2}$$

where the second line is just a trivial specialization of the first, to the case $\vec{r} = \vec{r}_1$ and $\vec{r}' = \vec{r}_2$, and we anticipate that $r_2 = r_<$, and $r_1 = r_>$ (why?). Write a computer symbolic program which calculates, explicitly and numerically,

$$T_\ell\,, \qquad 0 \le \ell \le 20\,. \tag{3}$$

Show that the sum converges slowly, and that the result

$$\sum_{\ell=0}^{20} T_\ell \qquad \text{approximates only 77\% of the full result for} \qquad \frac{1}{|\vec{r}-\vec{r}'|}\,. \tag{4}$$

Then, variously, calculate the first 201 terms $0 \le \ell \le 200$ and show that the full result for $1/|\vec{r}-\vec{r}'|$ is obtained to better than 99% agreement, or, explore convergence acceleration methods by which you can accelerate the convergence of the sum over ℓ, given only the first twenty (or so) terms.

Task 2 (70 + 30 EXTRA!!! points)
We have defined the elements of the quadrupole tensor of a charge distribution in spherical coordinates,

$$q_{2m} = \int \mathrm{d}^3 r\, r^2\, \rho(\vec{r})\, Y^*_{2m}(\theta,\varphi)\,, \qquad m = -2,-1,0,1,2\,. \tag{5}$$

We have also defined the elements of the quadrupole tensor in Cartesian coordinates,

$$Q_{ij} = \int \mathrm{d}^3 r\, \left(3 r_i\, r_j - \delta_{ij}\, \vec{r}^{\,2}\right)\, \rho(\vec{r})\,, \qquad i,j = 1,2,3\,. \tag{6}$$

We had convinced ourselves that the tensor Q_{ij} is traceless and symmetric, and therefore has only five independent components, say, Q_{11}, Q_{22}, Q_{12}, Q_{13}, and Q_{23}. Show that the conversion matrix looks similar to (no guarantee in regard to typos!)

$$\begin{pmatrix} q_{2\,-2} \\ q_{2\,-1} \\ q_{2\,0} \\ q_{2\,1} \\ q_{2\,1} \end{pmatrix} = \begin{pmatrix} \frac14\sqrt{\frac{5}{6\pi}} & -\frac14\sqrt{\frac{5}{6\pi}} & \frac{\mathrm{i}}{2}\sqrt{\frac{5}{6\pi}} & 0 & 0 \\ 0 & 0 & 0 & \frac12\sqrt{\frac{5}{6\pi}} & \frac{\mathrm{i}}{2}\sqrt{\frac{5}{6\pi}} \\ -\frac14\sqrt{\frac{5}{\pi}} & -\frac12\sqrt{\frac{5}{\pi}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac12\sqrt{\frac{5}{6\pi}} & \frac{\mathrm{i}}{2}\sqrt{\frac{5}{6\pi}} \\ \frac14\sqrt{\frac{5}{6\pi}} & -\frac14\sqrt{\frac{5}{6\pi}} & -\frac{\mathrm{i}}{2}\sqrt{\frac{5}{6\pi}} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{13} \\ Q_{23} \end{pmatrix} \tag{7}$$

For the 30 extra points: Show that the conversion matrix is invertible, and calculate its inverse, possibly with the help of a computer symbolic program.

The tasks are due Thursday, 11–NOV–2021.